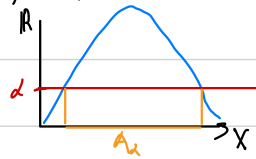
Measure Theory with Ergodic Horizons
Lecture 19

Properties of integrable functions.

Chebyshev's inequality. For each $f \in L^1(X,\mu)$ and $\alpha \in (0,\infty]$, we have

$$\mu(\underbrace{\{x \in X : |f(x)| \geq \alpha\}}_{A_\alpha}) \leq \frac{1}{\alpha}\|f\|_1.$$



In particular, $|f| < \infty$ a.e.

Proof. $\alpha \cdot \mu(A_\alpha) = \int_{A_\alpha} \alpha \, d\mu \leq \int_{A_\alpha} |f| \, d\mu \leq \int_X |f| \, d\mu = \|f\|_1$ $\square$

Cor. For each $f \in L^1(X,\mu)$, the support $\mathrm{supp}(f) := \{x \in X : f(x) \neq 0\}$ is σ-finite for μ, i.e. $\mathrm{supp}(f) = \bigcup_{n \in \mathbb{N}} B_n$ where each B_n has finite measure.

Proof. Let $B_n := A_{1/n} = \{x \in X : |f(x)| \geq \frac{1}{n}\}$, $n \geq 1$, then $\mathrm{supp}(f) = \bigcup_{n \in \mathbb{N}} B_n$ and $\mu(B_n) \leq \frac{1}{n} \cdot \|f\|_1 < \infty$. $\square$

Now we discuss properties of $\int |f| \, d\mu$ viewing $A \mapsto \int_A |f| \, d\mu$ as a measure $\mu_{|f|}$.

99% boundedness. For any $f \in L^1(X,\mu)$ and $\varepsilon > 0$, there is a set $X' \subseteq X$ such that $f|_{X'}$ is bounded and $\int_{X'} f \, d\mu \approx_\varepsilon \int_X f \, d\mu$.

Proof. Let $X_n := \{x \in X : |f(x)| \leq n\}$, hence $X = \biguplus_{n \in \mathbb{N}} X_n$ (excluding a null set, which we ignore), so $\mu_{|f|}(X_n) \nearrow \mu_{|f|}(X)$, i.e. $\int_{X_n} |f| \, d\mu \nearrow \int_X |f| \, d\mu < \infty$, hence $\int_{X \setminus X_n} |f| \, d\mu < \varepsilon$ for large enough $n \in \mathbb{N}$.

Thus, $\left| \int_{X_n} f \, d\mu - \int_X f \, d\mu \right| = \left| \int_{X \setminus X_n} f \, d\mu \right| \leq \int_{X \setminus X_n} |f| \, d\mu < \varepsilon$. $\square$

Def. Let $(X, \mathcal{B})$ be a measurable space. For measures μ, ν on $(X, \mathcal{B})$, we say that μ is absolutely continuous with respect to ν, and write $\mu \ll \nu$, if the null sets of ν are also null sets of μ, i.e. if $\nu(B) = 0$ then $\mu(B) = 0$.
We say that μ and ν are equivalent if $\mu \ll \nu$ and $\nu \ll \mu$, i.e. they have the same null sets.

Example. For each $f \in L^+(X, \mu)$, $\mu_f \ll \mu$ because if $\mu(B) = 0$ then $\mu_f(B) = \int_B f \, d\mu \leq \infty \cdot \mu(B) = \infty \cdot 0 = 0$.

The following explains the choice of the term "absolutely continuous":

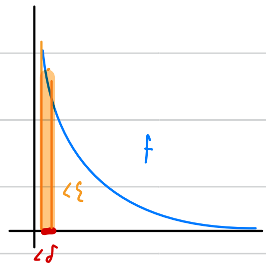
Prop. Let $(X, \mathcal{B})$ be a measurable space and μ, ν measures on it where μ is finite. Then $\mu \ll \nu$ if and only if $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $\nu(B) < \delta \Rightarrow \mu(B) < \varepsilon$ for all $B \in \mathcal{B}$.

Proof. $\Leftarrow$: This is trivial. If $\nu(B) = 0$ then it follows that $\mu(B) < \varepsilon$ for each $\varepsilon > 0$.

$\Rightarrow$. We show the contrapositive. Suppose $\exists \varepsilon > 0$ such that $\forall \delta > 0$ $\exists B_\delta \in \mathcal{B}$ such that $\nu(B_\delta) < \delta$ but $\mu(B_\delta) \geq \varepsilon$. This says that the collection $\mathcal{B}_\varepsilon := \{B \in \mathcal{B} : \mu(B) \geq \varepsilon\}$ contains arbitrarily ν-small sets (and is closed under ctbl unions). Thus, by our first application of Borel-Cantelli (Lecture 7), there is a ν-vanishing sequence $(B_n) \subseteq \mathcal{B}_\varepsilon$, i.e. $\bigcap_{n \in \mathbb{N}} B_n$ is ν-null. But $\mu(B_n) \geq \varepsilon$, so by downward continuity of finite measures, we have $\mu(\bigcap_{n \in \mathbb{N}} B_n) \geq \varepsilon$, hence $\mu \not\ll \nu$. $\square$

Applying this to $\mu_{|f|} \ll \mu$, we get:

Cor. For each $f \in L^1(X, \mu)$ we have $\forall \varepsilon > 0$ $\exists \delta > 0$ such that
$$\mu(B) < \delta \Rightarrow \int_B |f| \, d\mu < \varepsilon$$
for all μ-measurable $B \subseteq X$.

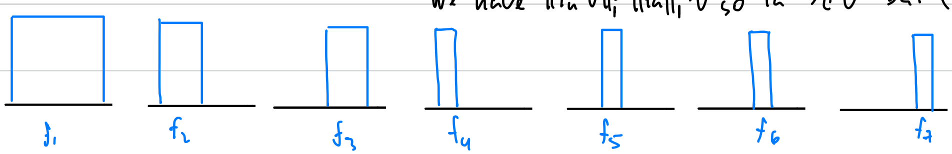


Convergence in measure.

Recall the following example where $(f_n) \subseteq L^1(X, \mu)$ converges in L^1-norm but not ptwise.

$f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,1/2]}$, $f_6 = \chi_{[1/2,3/4]}$, $f_7 = \chi_{[3/4,1]}$, and in general, $f_n = \chi_{[j/2^k, (j+1)/2^k]}$ where $n = 2^k + j$ with $0 \leq j < 2^k$.

We have $\|f_n - 0\|_1 = \|f_n\|_1 \to 0$ so $f_n \to_{L^1} 0$ but (f_n) doesn't converge pointwise.



However, there is a subsequence, e.g. $(f_{2n})_{n\in\mathbb{N}}$, that converges to 0 a.e.
Turns out this is always true:

Theorem. Let $f_n, f \in L^1(X,\mu)$. If $f_n \to_{L^1} f$ then $f_{n_k} \to f$ a.e. for some subsequence.

To prove this, we will study an intermediate notion of convergence, which is useful in its own right: convergence in measure.

Def. Let (X,μ) be a measure space and $f, g : X \to \overline{\mathbb{R}}$ μ-measurable functions. For each $\alpha > 0$, put

$$\Delta_\alpha(f,g) := \{x \in X : |f(x) - g(x)| \geq \alpha\}$$

$$\delta_\alpha(f,g) := \mu(\Delta_\alpha(f,g)).$$

Note. For μ-measurable sets $A, B \subseteq X$, $\Delta_\alpha(\mathbb{1}_A, \mathbb{1}_B) = A \triangle B$ for all $\alpha \in (0,1]$. Also $\delta_\alpha(\mathbb{1}_A, \mathbb{1}_B) = \mu(A \triangle B) = d_\mu(A,B)$ is our usual pseudo-metric, for all $\alpha \in (0,1]$.

The functions δ_α are not even pseudo-metrics because the triangle inequality fails: Let $f \equiv 0$, $g \equiv 1$, $h \equiv 2$, then $\delta_2(f,g) = 0 = \delta_2(g,h)$ but $\delta_2(f,h) = \mu(X)$.
However, the following "additive" version of triangle inequality holds:

Prop (Quasi-$\triangle$-inequality for δ_α). For all $\alpha, \beta > 0$, and f, g, h μ-measurable functions, we have $\Delta_{\alpha+\beta}(f,h) \subseteq \Delta_\alpha(f,g) \cup \Delta_\beta(g,h)$. In particular:

$$\delta_{\alpha+\beta}(f,h) \le \delta_\alpha(f,g) + \delta_\beta(g,h).$$

Proof. For each $x \in X$,

because $|f(x)-g(x)| + |g(x)-h(x)| \geq |f(x)-h(x)|$

$$\begin{aligned} x \in \Delta_{\alpha+\beta}(f,h) \iff |f(x)-h(x)| \geq \alpha+\beta &\implies |f(x)-g(x)| + |g(x)-h(x)| \geq \alpha+\beta \\ &\implies |f(x)-g(x)| \geq \alpha \text{ or } |g(x)-h(x)| \geq \beta. \\ &\iff x \in \Delta_\alpha(f,g) \cup \Delta_\beta(g,h). \end{aligned}$$

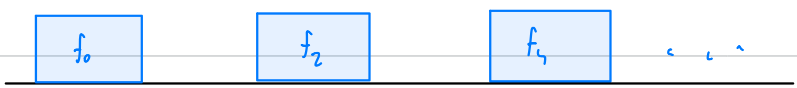
$\square$

Def. Let (f_n), f be μ-measurable functions. We say that (f_n) *converges to f in measure*, and write $f_n \to_\mu f$, if $\delta_\alpha(f_n, f) \to 0$ for each $\alpha > 0$, i.e. for each $\alpha > 0$,

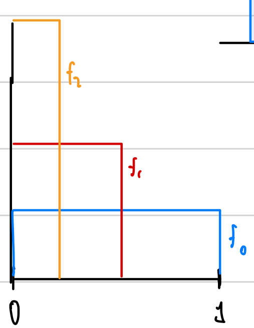
$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \alpha\}) \to 0 \quad \text{as } n \to \infty.$$

Examples.

(a) Let $f_n := \frac{1}{2}\mathbb{1}_{[n,n+1)}$, then $f_n \to 0$ pointwise, but not in L^1 (because $\|f_n - 0\|_1 = \frac{1}{2}$ $\forall n$) and not in measure (because $\delta_{1/2}(f_n, 0) = 1$ $\forall n$).

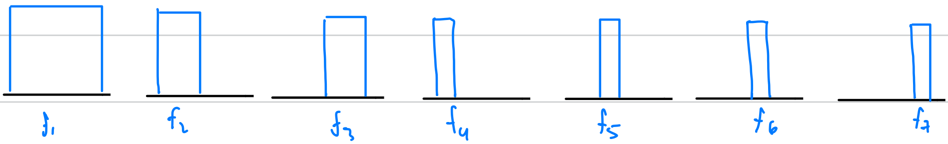


(b)



Let $f_n := n \cdot \mathbb{1}_{(0, 1/n]}$. Then $f_n \to 0$ pointwise and in measure (because $\delta_\alpha(f_n, 0) \leq \frac{1}{n}$ $\forall \alpha > 0$), but not in L^1 (because $\|f_n - 0\|_1 = 1$ $\forall n$).

(c) $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,1/2]}$, $f_6 = \chi_{[1/2,3/4]}$, $f_7 = \chi_{[3/4,1]}$, and in general, $f_n = \chi_{[j/2^k,\,(j+1)/2^k]}$ where $n = 2^k + j$ with $0 \le j < 2^k$.



Then (f_n) doesn't converge at any point, but $f_n \to_{L^1} 0$ (because $\|f_n - 0\|_1 = 2^{-\lfloor \log_2 n \rfloor} \to 0$) and $f_n \to_\mu 0$ (because $d_\mu(f_n, 0) = 2^{-\lfloor \log_2 n \rfloor} \to 0$). We will see next time that in general, L^1-convergence $\Rightarrow$ convergence in measure.